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ANALYTICAL METHODS OF SOLUTION OF CONJUGATED PROBLEMS IN CONVECTIVE HEAT TRANSFER

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Аннотация—В данной работе представлены методы решения задач конвективного теплообмена с учетом распространения тепла в твердом теле, соприкасающемся с движущейся средой. Этит методы названы методами решения сопряженных задач.

В частности, в работе рассматривается теплообмен при ламинарном течении жидкости в круглой и плоской трубах с учетом диссипации механической энергии. Кроме того, рассматриваются как стационарные, так и нестационарные задачи теплообмена при обтекании пластины сжимаемой жидкостью. При этом теплоперенос в жидкости во всех случаях рассматривается во взаимосвязи с переносом тепла в стенке твёрдого тела.

На основе полученного анализа решения вводится новый критерий, который характерезует влияние теплофизических свойств стенки на процесс теплообмена. Для иллюстрации рассматривается несколько конкретных примеров.

NOMENCLATURE

T,	temperature;
$\theta = [T/\chi_1(\infty)]$,	dimensionless temperature (internal problem);
$h = (T/T_\infty)$,	dimensionless temperature (external problem);
χ_1	temperature at the solid–fluid (gas) interface;
$\chi = [\chi_1/\chi_1(\infty)]$,	dimensionless temperature at the conjugation boundary (internal problem);
$\chi = (\chi_1/T_\infty)$,	dimensionless temperature at the solid-gas interface (external problem);
ψ_1,	temperature of external tube surface;
$\psi = [\psi_1/\chi_1(\infty)]$,	dimensionless temperature of external tube surface;
x, y_2, z, r, y_1,	spatial coordinates;
$\xi = (z/R)$, $\eta = (r/R)$,	dimensionless spatial coordinates (internal problem);
$y = (y_2/b)$, $\xi = (x/b)$,	dimensionless spatial coordinates (external problem);
$\tilde{h}$,	enthalpy;
R,	tube radius, $R_1 - R$, thickness of tube wall;
b,	plate thickness;
λ,	thermal conductivity;
c,	heat capacity;
σ,	Prandtl number;
$P_e = (v_0R/a)$,	Péclét number;
M,	Mach number;
α,	parameter of generalized Fourier sine transformation;
${}_1F_1(\alpha_1\beta_1 x)$,	confluent hypergeometric function;

h_{00}, retardation temperature;
h_e, recovery temperature;
v_1, mean velocity;

$$v_0 = \frac{3v_1}{2};\ A_1 = \frac{H}{16};\ F_1(\eta) = 1 + \beta\frac{H}{3}(1 - \eta)$$

for a plane tube;

$$v_0 = 2v_1;\ A_2 = \frac{H}{12};\ F_2(\eta) = 1 - \beta\frac{H}{4}\ln\eta$$

for a circular tube;

$$\beta = \frac{\lambda}{\lambda_f};\quad \delta = \frac{R_1}{R};$$

$$H = \frac{4v_0^2\,Pr}{c\varkappa_1(\infty)};\quad y_1 = y\sqrt{(R_{e\,\infty})};$$

$$Re = \frac{u\,.\,b}{\nu};\quad Re_\xi = \frac{ux}{\nu};\quad k = \frac{c_p}{c_v};$$

$$B = \frac{2Nu(\xi)}{\sqrt{(Re_\xi)}};\quad \bar{h}_w = \frac{h_w - h_l}{h_e - h_b};$$

$$\zeta = 0{\cdot}5\xi^{-0.5}\int_0^{y_1}\rho\,dy.$$

Subscripts and Superscripts

∞, refers to main flow parameters;
f, characteristic of a solid;
w, refers to the surface in a gas flow;
$m = 1, 2$; $m = 1$, a plane tube; $m = 2$, a circular tube.

CONVECTIVE heat transfer phenomena are of great practical importance in many fields of modern technology (missile and aircraft engineering, atomic power engineering).

Heat transfer problems in present-day high-speed aeroplanes involving considerable heating of the aerodynamic construction have become of particular significance.

A great deal of research efforts, both experimental and theoretical, has been devoted to these problems [1–6].

All these works are characterized by a general way of treatment: first, the problem is solved for the temperature distribution in the boundary layer of the main stream under the prescribed conditions at the surface, and the heat transfer coefficient is calculated. Next comes consideration of the heat transfer process in the solid. At the solid–fluid interface there are formulated the so-called third-kind boundary conditions involving the heat transfer coefficient calculated beforehand.

Thus, there is made an attempt to describe a complexity of heat transfer processes between the solid and the main stream by a single heat-transfer coefficient (many empirical and semi-empirical formulae have been obtained for its determination).

In such a statement the mutual thermal effect of the solid and the fluid is not allowed for, i.e. heat transfer process appears to be independent of the solid properties (thermophysical properties, dimensions, source distribution, etc.).

This statement does not seem to be physically strict. It is worth noting that, as it has become known nowadays [5, 6], the third-kind boundary conditions are not valid for many cases, since they lead to contradictory or even physically senseless results [5, 6].

Consideration of convective heat problems as conjugated problems [7] appears to be physically more strict, i.e. the energy equations for fluid and solid are attacked together with those of hydrodynamics; the temperatures and heat fluxes at the solid–fluid interface being considered equal (the fourth-kind boundary conditions are formulated).

This is the case when the mutual thermal effect of the solid body and the fluid is already taken into account contrary to the alternative statement.

At present a number of publications devoted to the solution of conjugated problems [7–9] are available.

The common points in them are as follows: only particular problems have been solved by various methods that required physically not warranted simplifications in their statement; neither new physical effects that should be expected have been obtained nor general methods of the conjugated problems have been developed.

Recently [10, 11] general methods for both internal and external conjugated heat transfer problems have been developed. These allowed analytical solutions to be obtained in a rather general statement for the cases of real physical phenomena as well as new physical effects.

The present paper reports two general solution method for internal and external conjugated problems of convective heat transfer.

1. INTERNAL CONJUGATED PROBLEMS

The general method developed for internal heat transfer conjugated problems is based on reducing the problem to a singular integral equation for the unknown temperature of the surface in a flow [11]. The method permits exact solutions for the case of both steady- and unsteady-state heat transfer in laminar and turbulent flows to be obtained.

The present paper is restricted (because of the lack of space) to consideration of a steady-state heat transfer problem with laminar forced convection in circular and plane tubes for the developed Poiseuille velocity distribution with allowance for mechanical energy dissipation.

1. Mathematically the problem reduces to a solution of dimensionless equations for a fluid

$$Pe(1-\eta^2)\frac{\partial\theta_m}{\partial\xi} = \frac{1}{\eta^{m-1}}\frac{\partial}{\partial\eta}\left(\eta^{m-1}\frac{\partial\theta_m}{\partial\eta}\right) + H\eta^2; \quad \begin{matrix} 0 \leqslant \eta \leqslant 1 \\ 0 \leqslant \xi \leqslant \infty \end{matrix} \tag{1}$$

with the boundary conditions

$$\theta_m|_{\xi=0} = \theta_{m_0} \qquad \theta_m|_{\xi\to\infty} = A_m(1-\eta^4)+1; \tag{2}$$

$$\left.\frac{\partial\theta_m}{\partial\eta}\right|_{\eta=0} = 0 \qquad \theta_m|_{\eta=1} = \chi_m(\xi); \tag{3}$$

and of the equation for the solid

$$\frac{\partial^2\theta_{fm}}{\partial\xi^2} + \frac{1}{\eta^{m-1}}\frac{\partial}{\partial\eta}\left(\eta^{(m-1)}\frac{\partial\theta_{fm}}{\partial\eta}\right) = 0 \quad \begin{matrix} 1 \leqslant \eta \leqslant \delta \\ 0 \leqslant \xi < \infty \end{matrix} \tag{4}$$

with the boundary conditions

$$\theta_{fm}|_{\xi=0} = \theta_{fm_0} \qquad \theta_{fm}|_{\xi\to\infty} = F_m(\eta) \tag{5}$$

$$\theta_{fm}|_{\eta=1} = \chi_m(\xi); \qquad \theta_{fm}|_{\eta=\delta} = \psi_m(\xi). \tag{6}$$

At the solid–fluid interface the conjugation conditions are

$$\theta_m|_{\eta=1} = \theta_{fm}|_{\eta=1} = \chi_m(\xi); \tag{7}$$

$$\beta\left.\frac{\partial\theta_m}{\partial\eta}\right|_{\eta=1} = \left.\frac{\partial\theta_{fm}}{\partial\eta}\right|_{\eta=1}. \tag{8}$$

Note, that the conditions at $\xi\to\infty$ in equations (2) and (5) derive from a solution of the problem stated at $\xi\to\infty$.

2. Solutions (1)–(3) are easy to obtain by applying the superposition principle in the form

$$\left.\begin{aligned} \theta_m = {} & A_m(1-\eta^4) - \sum_{n=1}^{\infty} A_{nm}R_{nm}(\eta)\int_0^{\xi}\chi(\xi_1) \\ & \times \exp[-b_{nm}(\xi-\xi_1)]\,d\xi_1 + \sum_{n=1}^{\infty} B_{nm}R_{nm}(\eta)\exp[-b_{nm}\xi]; \end{aligned}\right\} \tag{9}$$

where

$$A_{nm} = -\frac{\int_0^1 R_{nm}(\eta_1)\eta_1^{m-1}(1-\eta_1^2)\,d\eta_1}{\int_0^1 R_{nm}^2(\eta_1)\eta_1^{m-1}(1-\eta_1^2)\,d\eta_1} = \frac{2}{P_{nm}\left(\dfrac{\partial R_{nm}}{\partial P_{nm}}\right)_{\eta=1}};$$

$$B_{nm} = \frac{2\int\limits_0^1 R_{nm}(\eta_1)\,\eta_1^{m-1}(1-\eta_1^2)\,[\theta_0 - A_m(1-\eta_1^4)]\,d\eta_1}{[(\partial R_{nm}/\partial R_{nm})\,.\,(\partial R_{nm}/\partial\eta)]_{\eta=1}};$$

$$R_{nm}(\eta) = \exp\,[-0{\cdot}5\,P_{nm}\eta^2]\,{}_1F_1\,[\alpha_m, \beta_m, P_{nm}\,\eta^2];$$

$$\alpha_1 = \frac{1-P_{n_1}}{4};\quad \alpha_2 = \frac{2-P_{n2}}{4};\quad \beta_1 = 0{\cdot}5;\quad \beta_2 = 1;\quad P_{nm}^2 = b_{nm}.$$

P_{nm} are the roots of the characteristic equation

$$_1F_1(\alpha_m, \beta_m, P_m) = 0 \tag{10}$$

which can be found, for example, in [13].

3. The boundary-value problem solution for the solid (4)–(6) is found by the generalized Fourier sine transformation

$$\theta_{f1} = \theta_{f10} + \lim_{\mu\to 0}\frac{1}{\pi} < \int\limits_0^\infty \left\{\left[\chi(\alpha) - \frac{\theta_{f10}}{\alpha}\right] \times \operatorname{sh}\alpha(b-\eta) + \left[\psi_1(\alpha) - \frac{\theta_{f10}}{\alpha}\right]\operatorname{sh}\alpha(\eta-1)\right\}$$
$$\times\frac{\exp\,[-\mu\alpha]}{\operatorname{sh}\alpha(b-1)}\sin\alpha\xi\,d\alpha >; \tag{11}$$

$$\theta_{f2} = 1 - 0{\cdot}25\,\beta H\ln\eta + \lim_{\mu\to 0}\frac{1}{\pi} < \int\limits_0^\infty\left\{\left[\chi(\alpha) - \frac{1}{\alpha}\right]\right.$$
$$\times\frac{K_0(\alpha\delta)\,I_0(\alpha\eta) - I_0(\alpha\delta)\,K_0(\alpha\eta)}{I_0(\alpha)\,K_0(\alpha\delta) - I_0(\alpha\delta)\,K_0(\alpha)} - \int\limits_1^\delta G(\eta,\eta_1,\alpha)$$
$$\times\,[0{\cdot}25\,\beta H\ln\eta_1 - \psi_2(\infty)]\,d\eta_1\,\}\exp\,[-\mu\alpha]\sin\alpha\xi\,d\alpha >;$$
$$G(\eta,\eta_1,\alpha) = \begin{cases}\dfrac{\alpha[K\,(\alpha\eta_1)\,I_0(\alpha\delta) - K_0(\alpha\delta)\,I_0(\alpha\eta_1)]}{I^0(\alpha\delta)}\,I_0(\alpha\eta); & 1 \leqslant \eta \leqslant \eta_1;\\[2ex] \dfrac{\alpha[K_0(\alpha\eta)\,I_0(\alpha\delta) - K_0(\alpha\delta)\,I_0(\alpha\eta)]}{I_0(\alpha\delta)}\,I_0(\alpha\eta_1); & \eta_1 \leqslant \eta \leqslant \delta.\end{cases} \tag{12}$$

Performing substitution of (9), (11) and (12) into the conjugation conditions (8), written for the transforms, we obtain the equation for the transform of the unknown function $\chi_m(\alpha)$

$$\varphi_m(t)\,a_m(t) - b_m(t)\int\limits_0^\infty\frac{\varphi_m(\tau)}{\tau - t}\,d\tau = f_m(t); \tag{13}$$

where,

$$\varphi_m(t) = \chi_m(\alpha);\quad t = \alpha^2.$$

$$a_1(t) = \sum_{n=1}^\infty\frac{A_{n_1}\,R'_{n_1}(1)\,b_{n_1}^2}{b_{n_1}^2 + t} - \surd(t)\beta^{-1}\operatorname{cth}\surd(t)\,(\delta - 1);$$

$$b_1(t) = 0{\cdot}5\beta \sum_{n=1}^{\infty} \frac{\sqrt{(t)}\, A_{n_1} R_n'(1)\, b_{n_1}}{b_{n_1}^2 + t};$$

$$f_1(t) = -\frac{\beta H}{3\sqrt{t}} + \beta \sum_{n=1}^{\infty} B_{n_1} R_{n_1}'(1) \frac{\sqrt{t}}{b_{n_1}^2 + t} - \theta_{f_{10}} \operatorname{cth} \sqrt{(t)}\,(\delta - 1) + \frac{\theta_{f_{10}} - \sqrt{(t)}\,\psi_1(t)}{\operatorname{sh} \sqrt{(t)}\,(\delta - 1)};$$

$$a_2(t) = \sum_{n=1}^{\infty} \frac{A_{n_2} R_{n_2}'(1)\, b_{n_2}^2}{b_{n_2}^2 + t} + \frac{\sqrt{t}[K_0(\delta\sqrt{t})\, I_1(\sqrt{t}) + I_0(\delta\sqrt{t})\, K_1(\sqrt{t})]}{\beta[I_0(\sqrt{t})\, K_0[\sqrt{(t)}\delta] - I_0(\sqrt{t}\delta)\, K_0(\sqrt{t})]};$$

$$b_2(t) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{A_{n_2} R_{n_2}'(1)\, b_{n_2} \sqrt{t}}{b_{n_2}^2 + t};$$

$$f_2(t) = \sum_{n=1}^{\infty} \frac{B_{n_2} R_{n_2}'(1) \sqrt{t}}{b_{n_2}^2 + t} + \frac{1}{\beta} \Bigg\{ \sqrt{t} \int_1^{\delta} G'(1, \rho_1, \sqrt{t})$$
$$\times \left[\frac{H\beta \ln \rho_1}{4} - \psi_2(\infty)\right] d\rho_1 + \frac{K_0(\delta\sqrt{t})\, I_1(\sqrt{t}) + I_0(\delta\sqrt{t})\, K_1(\sqrt{t})}{I_0(\sqrt{t})\, K_0(\delta\sqrt{t}) - I_0(\delta\sqrt{t})\, K_0(\sqrt{t})}.$$

4. Equation (13) is a singular integral equation containing the Cauchy formulation (14). To solve it, we take advantage of the concept of analytical extension to the complex region and reduce equation (13) to the Riemann boundary-value problem with discontinuity coefficients. On introduction of the analytical function

$$\hat{\Phi}(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)\, d\tau}{\tau - z}; \tag{14}$$

where contour L is a positive part of the real axis. Using Sokhotsky–Plemel's formulae, we arrive at

$$\varphi(t) = \Phi^+(t) - \Phi^-(t); \tag{15}$$

$$\frac{1}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \Phi^+(t) + \Phi^-(t). \tag{16}$$

Substitution of (15) and (16) into (13) yields

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t); \tag{17}$$

where

$$G(t) = \frac{a(t) + b(t)\pi i}{a(t) - b(t)\pi i}; \quad g(t) = \frac{f(t)}{a(t) - b(t)\pi i}. \tag{17}$$

The nonhomogeneous boundary-value Riemann problem (15) may be solved in a general form, if the problem index is not negative. It is easy to demonstrate that for the present case the index is zero [11] and solution (13) assumes the form

$$\chi(\alpha) = 0{\cdot}5\, g(\alpha^2)\, [1 + G^{-1}(\alpha^2)] + \sqrt{[G(\alpha^2)]}$$
$$\times \exp[\Gamma(\alpha^2)] \times [1 - G^{-1}(\alpha^2)]$$

$$\times \frac{1}{\pi i} \int_0^{\infty} \frac{g(y^2)\, y\, dy}{\sqrt{(G(y^2))}\,(y^2 - \alpha^2) \exp[\Gamma(y^2)]}; \tag{18}$$

where

$$\Gamma(t) = \frac{1}{2\pi i} \int_L \frac{\ln G(\tau)}{\tau - t} d\tau.$$

The desired temperatures θ_m and θ_{fm} are found from equations (9), (11) and (12) by considering (18) and the relationship in reference [12].

$$\chi_m(\xi) = (1/\pi) \lim_{\mu \to 0} \int_0^\infty \chi_m(\alpha) \exp[-\mu\alpha] \sin \alpha\xi \, d\alpha.$$

EXTERNAL CONJUGATED PROBLEMS

The general method developed for external conjugated problems of convective heat transfer is based upon series expansion with respect to powers of the parameter of the generalized Fourier sine transform of the temperature profile for flow past a surface.

This method has permitted the authors to obtain the solutions for both steady- and unsteady-state heat transfer problems for a plate in liquid or gas flows; to allow for radiation effect [11] as well as injection (or suction); in addition, by the method suggested the authors have succeeded to obtain new criterion κ for a mutual thermal effect of the solid body and the liquid.

The present paper is restricted to consideration of a steady-state problem for a plate (length L, thickness b, thermal conductivity λ_f) in a longitudinal gas flow of constant velocity u_∞ and temperature T_∞ at infinity; the internal surface temperature is assumed to be constant (a similar solution is obtained for the case when the flow is prescribed over the internal surface).

Gas velocity is assumed to be temperature-dependent and to follow a linear law; the Prandtl number is considered to be constant.

1. Mathematically the problem reduces to a solution of dimensionless equations for the gas

$$\frac{\partial^2 h}{\partial \zeta^2} + \sigma\varphi \frac{\partial h}{\partial \zeta} - 2\sigma\varphi' \xi \frac{\partial h}{\partial \xi} = -\frac{\sigma(k-1)}{4} \times M_\infty^2 [\varphi''(\zeta)]^2 \qquad (19)$$

with the boundary conditions

$$h|_{\zeta=0} = h_e + \chi(\xi); \quad h|_{\zeta \to \infty} = 1 \qquad (20)$$

and the equation for the solid

$$\frac{\partial^2 h_f}{\partial \xi^2} + \frac{\partial^2 h_f}{\partial y^2} = 0 \qquad (21)$$

with the boundary conditions

$$h_f\Big|_{\xi \to 0} = h_{00} \qquad \frac{\partial h_f}{\partial \xi}\Big|_{\xi \to \infty} = 0 \qquad (22)$$

$$h_f|_{y=0} = h_e + \chi(\xi); \quad h_f|_{y=-1} = h_b. \qquad (23)$$

At the body-fluid interface the conjugation conditions are given as

$$h|_{\zeta=0} = h_f|_{y=0}; \qquad \varkappa\xi^{-0.5} \frac{\partial h}{\partial \zeta}\Big|_{\zeta=0} = \frac{\partial h_f}{\partial y}\Big|_{y=0}. \qquad (24)$$

Later on, the unknown function $\chi(\xi)$ is determined from the second conjugation condition (24) for the flows.

As is known, in case of a linear dependence of viscosity upon temperature, the hydrodynamic problem is autonomous (i.e. functions $\varphi(\zeta)$, $\varphi'(\zeta)$, $\varphi''(\zeta)$ are known, this is the ordinary Blasius profile [3]).

The solution of problem (19) and (20) for the gas is to be sought in the form

$$h(\zeta_1\xi) = 1 + \frac{k-1}{8} M_\infty^2 \vartheta(\zeta) + \bar{h}(\zeta_1\xi); \qquad (25)$$

where

$$\vartheta(\zeta) = 2\sigma \int_\zeta^\infty [\varphi''(\zeta_1)]^\sigma \, d\vartheta \int_0^{\zeta_1} [\varphi''(\zeta_2)]^{2-\sigma} d\vartheta_2; \qquad (26)$$

and $\bar{h}$ satisfies the boundary-value problem

$$\frac{\partial^2 \bar{h}}{\partial \zeta^2} + \sigma\varphi \frac{\partial \bar{h}}{\partial \zeta} - 2\sigma\varphi' \xi \frac{\partial \bar{h}}{\partial \xi} = 0; \qquad (27)$$

$$\bar{h}|_{\zeta=0} = \chi(\xi); \qquad \bar{h}|_{\zeta \to \infty} = 0. \qquad (28)$$

The generalized Fourier sine transformation

$$u_s(\alpha) = \lim_{\mu \to 0} \int_0^\infty u(\xi) \sin \alpha\xi \exp(-\mu\xi) \, d\xi \qquad (29)$$

yields the transformed equations (27) and (28)

$$\frac{\partial^2 \bar{h}_s}{\partial \zeta^2} + \sigma\varphi \frac{\partial \bar{h}_s}{\partial \zeta} + 2\sigma\varphi'(\zeta) \left[\bar{h}_s + \alpha \frac{\partial \bar{h}_s}{\partial \alpha}\right] = 0 \qquad (30)$$

$$\bar{h}_s|_{\zeta=0} = \chi_s(\alpha); \quad \bar{h}|_{\zeta\to\infty} = 0. \tag{31}$$

The solution of equations (30) and (31) is to be found in a series form

$$\bar{h}_s(\zeta, \alpha) = \sum_k C_k Z_{vk} \alpha^{vk}; \tag{32}$$

where $Z(\zeta)$ satisfies the boundary-value problem

$$Z''(\zeta) + \sigma\varphi Z'(\zeta) + 2\sigma\varphi'(1+\nu)Z(\zeta) = 0; \tag{33}$$

$$Z|_{\zeta=0} = 1 \quad Z|_{\zeta\to\infty} = 0. \tag{34}$$

C_k, ν_k are constants to be determined later on. Here the transform of the unknown function is

$$\chi_s(\alpha) = \sum C_\nu \alpha^{\nu_k}. \tag{35}$$

The transformed solution (21)–(23) with allowance for (29) may be written as

$$h_{fs} = \frac{h_{00}}{\alpha} + \frac{\mathrm{sh}(\alpha)(1+y)}{\mathrm{sh}\,\alpha}\chi_s(\alpha)$$
$$- \frac{(h_{00} - h_e)\,\mathrm{sh}(\alpha)(1+y) + (h_b - h_{00})\mathrm{sh}\,\alpha y}{\alpha\,\mathrm{sh}\,\alpha}. \tag{36}$$

Differentiation of equations (25) and (36) and substitution of the results obtained into the second conjugation condition (24) for flows written for transforms yield the equation from which constants ν_k and C_k are determined

$$\varkappa \sum_k C_k A_k \alpha^{\nu_k + 0.5}\,\mathrm{sh}\,\alpha$$
$$= (h_b - h_{00})\,\mathrm{ch}\,\alpha + h_{00} - h_b + \alpha\sum_k C_k \alpha^{\nu_k}\mathrm{ch}\,\alpha;$$

where

$$A_k = Z_{\nu_k}(0)\frac{\Gamma(\nu_k+1)\sin[(\nu_k+1)\pi/2]}{\Gamma(\nu_k+1.5)\sin[(\nu_k+1.5)\pi/2]}$$

$$\varkappa = 0.5\frac{\lambda_\infty}{\lambda_f}\sqrt{Re_\infty}.$$

Performing expansion in power series of the functions entering into equation (37) and equating coefficients with like powers α give ν_k and C_k values determined by the formulae

$$\nu_k = 0.5(k-2): \quad k = 1,2\ldots; \quad j = 1,2\ldots;$$

$$\varkappa A_{k-1}C_{k-1} + \varkappa\sum_{n=1}^{(k-1/4)}\frac{A_{k-1-4n}C_{k-1-4n}}{(2n+1)!} - \sum_{n=1}^{[k/4]}\frac{C_{k-4n}}{(2n)!}; \quad k \neq 4j$$

$$\varkappa A_{k-1}C_{k-1} + \varkappa\sum_{n=1}^{(k-1/4)}\frac{A_{k-1-4n}C_{k-1-4n}}{(2n+1)!} - \sum_{n=1}^{[k/4]}\frac{C_{k-4n}}{(2n)!} + \frac{h_{00} - h_e}{(k/2)!}; \quad k = 4j.$$

Applying V_k and C_k values obtained and inverting by the formula

$$\alpha^\mu \leftrightarrow \frac{2}{\pi}\frac{\Gamma(\mu+1)}{\xi^{\mu+1}}\sin\left[\frac{(\mu+1)\pi}{2}\right]; \tag{38}$$

the desired temperature profile in a boundary layer is determined from equation (35) as

$$\begin{aligned}h(\zeta,\xi) = {} & 1 + [(k-1)/8]M_\infty^2\nu(\zeta) \\ & + (h_b - h_e)\int_\zeta^\infty[\varphi''(\zeta_1)]^\sigma\,\mathrm{d}\zeta_1\Big(\int_0^\infty[\varphi''(\zeta_1)]^\sigma\,\mathrm{d}\zeta_1\Big)^{-1} \\ & + \varkappa Z'_{-1}(0)(h_b - h_e) \\ & < [\varphi''(\zeta)/\varphi''(0)]^\sigma\xi^{-0.5} + 0.25Z_{1.5}(\zeta)\xi^{-2.5} \\ & + 0.25\varkappa Z'_{1.5}(0)Z_2(\zeta)\xi^{-3} \\ & + 0.25\varkappa^2 Z'_2(0)Z'_{1.5}(0)Z_{2.5}(\zeta)\xi^{-3.5} \\ & + (7.5\sqrt{2}/16\sqrt{\pi}) \\ & \varkappa^3 Z'_{2.5}(0)Z'_2(0)Z'_{1.5}(0)Z_3(\zeta)\xi^{-4} + \ldots.\end{aligned} \tag{39}$$

With $\zeta = 0$ the surface temperature in a gas flow is obtained from equation (39) as

$$\begin{aligned}h_w = h|_{\zeta=0} = h|_{y=0} = {} & h_b + \varkappa Z'_{-1}(0)(h_b - h_e) \\ & < \xi^{-0.5} + 0.25\xi^{-2.5} + 0.25\varkappa Z'_{1.5}(0)\xi^{-3} \\ & + 0.25\varkappa^2 Z'_2(0)Z'_{1.5}(0)\xi^{-3.5} \\ & + (7.5\sqrt{2}/16\sqrt{\pi}) \\ & \varkappa^3 Z'_{2.5}(0)Z'_2(0)Z'_{1.5}(0)\xi^{-4} + \ldots.\end{aligned} \tag{40}$$

Note, that functions Z_{V_k} in equations (39) and (40) with V_k found are determined as the boundary-value problem solution

$$Z''_{V_k} + \sigma\varphi Z'_{V_k} + 2\sigma\varphi'(1 + V_k) Z_{V_k} = 0 \qquad Z_{V_k}|_{\zeta=0} = 1 \quad Z_{V_k}|_{\zeta\to\infty} = 0. \tag{41}$$

The exact solutions are obtained only for two cases, when $V_0 = -1$, $V_1 = -0{\cdot}5$.

$$Z_{-1}(\zeta) = \int_{\zeta}^{\infty} [\varphi''(\zeta_1)]^{\sigma} d\zeta_1 \left(\int_0^{\infty} [\varphi''(\zeta_1)]^{\sigma} d\zeta_1\right)^{-1}; \tag{42}$$

$$Z_{-0\cdot5}(\zeta) = \left[\frac{\varphi''(\zeta)}{\varphi''(0)}\right]^{\sigma}. \tag{43}$$

With the other values of V_k the functions Z_{V_k} may be obtained only numerically which requires an asymptotic solution of equation (41) to be found first. The Blasius functions $\varphi(\zeta)$ and $\varphi'(\zeta)$ entering into equation (41) are not expressed analytically but for large $\zeta(\zeta \geqslant 3{\cdot}5)$ they may be represented as

$$\varphi(\zeta) \cong 2(\zeta - 0{\cdot}86); \quad \varphi'(\zeta) = 2; \quad \zeta \geqslant 3{\cdot}5. \tag{44}$$

Substitution of (44) into (41) produces the equation for large ζ

$$Z''_{V_k}(\zeta) + 2\sigma(\zeta - 0{\cdot}86) Z'_{V_k}(\zeta) + 4\sigma(1 + v_k) Z(\zeta) = 0 \qquad Z_{V_k}|_{\zeta\to\infty} = 0 \tag{45}$$

the solution of which is

$$Z_{v_k}|_{\zeta \geqslant 3\cdot5} = C \frac{d^{2v+1}}{d\zeta^{2v+1}} \{\exp[-0{\cdot}5(\zeta - 0{\cdot}86)^2]\}. \tag{46}$$

Numerical calculation is performed with allowance for (46), the constant C being determined at the end of the calculation at $\zeta = 0$. [11] reports results obtained by the finite-difference method. Note that functions Z_{v_k} at $v_k \geqslant 0$ are non-monotonous.

It should be emphasized that solutions (39) and (40) include the new criterion

$$\varkappa = 0{\cdot}5 \frac{\lambda_\infty}{\lambda_f} \sqrt{(Re_\infty)} = 0{\cdot}5 \frac{\lambda_\infty}{\lambda_f} \sqrt{(u_\infty b / V_\infty)}; \tag{47}$$

which comprises characteristics of both the flow and the body. At $\varkappa \to 0$ (that corresponds to an infinite heat conductivity of a solid or infinitesimal thickness of a plate $b \to 0$) there is obtained the known gas temperature profile in a flow at constant temperature past a solid body

$$h^*(\zeta, \xi| = \lim_{\varkappa\to 0} h(\zeta, \xi) = 1 + [(k-1)/8] M_\infty^2 \, v(\zeta) + (h_b - h_e) \int_{\zeta}^{\infty} [\varphi''(\zeta_1)]^{\sigma} d\zeta_1 \left(\int_0^{\infty} [\varphi''(\zeta_1)]^{\sigma} d\zeta_1\right)^{-1}. \tag{48}$$

Generally speaking, the criterion $\varkappa$ may take on values from 0 to ∞ (for a majority of real situations $\varkappa$ is of an order of unity); note that the extreme case $\varkappa \to \infty$ corresponds to thermal insulation of a body in a flow, and $\varkappa \to 0$ is characteristic of the case when use of Newton's law is possible with an error that may be found.

In all other cases ($0 < \varkappa < \infty$) the heat transfer coefficient may be only formally introduced. Hence the Nusselt number with regard for equation (40) is determined as

$$\begin{aligned} Nu(\xi) = {} & 0{\cdot}5 \sqrt{(Re_\xi} < 0{\cdot}664 \sqrt[3]{(Pr)} - \varkappa Z'_{-1}(0) [0{\cdot}25 Z'_{1\cdot5}(0) \xi^{-2\cdot5} \\ & + 0{\cdot}25 \varkappa Z'_2(0) Z'_{1\cdot8}(0) \xi^{-3} + 0{\cdot}25 \varkappa^2 Z'_{2\cdot5}(0) Z'_2(0) Z'_{1\cdot5}(0) \xi^{-3\cdot5} \\ & + (7{\cdot}5 \sqrt{2}/16 \sqrt{\pi}) \varkappa^3 Z'_{2\cdot5}(0) Z'_2(0) Z'_{1\cdot5}(0) \xi^{-y} + \ldots] > \\ & \times < 1 + \varkappa Z'_{-1}(0) [\xi^{-0\cdot5} + 0{\cdot}25 \xi^{-2\cdot5} + 0{\cdot}25 \varkappa Z'_{1\cdot5}(0) \xi^{-3} \\ & + 0{\cdot}25 \varkappa^2 Z'_d(0) Z'_{1\cdot5}(0) \xi^{-3\cdot5} + (7{\cdot}5 \sqrt{2}/16 \sqrt{\pi}) \varkappa^3 Z'_{2\cdot5}(0) Z'_2(0) Z'_{1\cdot5}(0) \xi^{-y} + \ldots]^{-1}. \end{aligned} \tag{49}$$

It is easy to see that at $\varkappa \to 0$ the well-known Nusselt numbers are obtained from (49)

$$Nu^* = \lim_{\varkappa \to 0} Nu = 0{\cdot}332\sqrt{(Re_\xi)}\,.\,\sqrt[3]{(Pr)}. \tag{50}$$

2. Consider two calculational examples by formulae derived in [11]

(a) Poor conducting material ($\lambda_f = 2{\cdot}52$ W/(mdeg): The plate is $b = 0{\cdot}01$m thick. The flow characteristics are $M_\infty = 3$; $T_b = 280°$K; $T_\infty = 223°$K $v_\infty = 9{\cdot}54 \times 10^{-6}$m^2/s; $\lambda_\infty = 2{\cdot}04 \times 10^{-2}$ W/mdeg; $U_\infty{}^{(1)} = 900$ m/s; $u_\infty{}^{(2)} = 400$ m/s. In this case $\varkappa^{(1)} = 3{\cdot}95$; $\varkappa^{(2)} = 2{\cdot}64$. By equations (40) and (49) there have been calculated the temperature of the surface and the Nusselt numbers. The results are presented in Figs. 1 and 2.

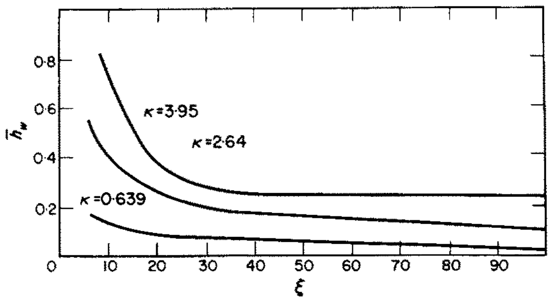

FIG. 1.

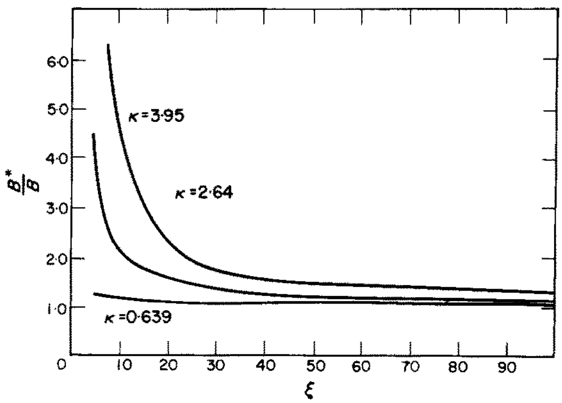

FIG. 2.

(b) Well conducting material ($\lambda_f = 15{\cdot}6$ W/(mdeg: In this case $\varkappa^{(1)} = 0{\cdot}639$. Calculational results similar to the first case are presented in Figs. 1 and 2.

From the graphs obtained it is seen that h_w and Nu may differ considerably from those usually used (at $\varkappa \to 0$). Thus, in case (a) at individual points Nu values differ several times from the ordinary Nu^*; the same refers to the temperature of the surface in a flow (since at $\varkappa \to 0$, $hw^* = h_b$). At the same time for a good conducting material h_w and Nu values do not differ very much (by not more than 25 per cent) from the corresponding values of h_w^* and Nu^* (for the case $\varkappa \to 0$).

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Abstract—The present paper presents solution methods of convective heat transfer problems which take into account heat propagation in the solid in contact with a moving fluid. The method is referred to as the solution of conjugated problems.

In particular, the paper treats heat transfer in laminar fluid flow in circular and planar tubes with allowance for the dissipation of mechanical energy. In addition, there are considered both steady-and unsteady-state heat transfer problems for flow of a compressible fluid past a plate. In all cases heat transfer in the fluid is discussed in relation to that in a solid wall.

On the basis of the analysis of the solution a new criterion is introduced which characterizes the effect of thermophysical properties of the wall on heat transfer. A few examples are considered for illustration purposes.

MÉTHODES ANALYTIQUES POUR DES PROBLÈMES CONJUGUÉS DE CONVECTION THERMIQUE

Résumé—Cet article concerne les méthodes de solution de problèmes de convection thermique en tenant compte de la propagation thermique dans un solide en contact avec un écoulement de fluide. La méthode s'inspire de la recherche de la solution des problèmes conjugués. En particulier, l'article traite le transfert thermique dans un écoulement de fluide laminaire à l'intérieur de tubes circulaires et plans avec admission d'énergie mécanique. De plus, on considère à la fois les problèmes de transfert thermique stationnaire et instationnaire pour une plaque dans un écoulement compressible. Dans tous les cas, le transfert de chaleur dans le fluide est discuté en relation avec celui dans une paroi solide. A partir de cette analyse, on introduit un nouveau critère qui caractérise l'effet des propriétés thermophysiques de la paroi sur le transfert thermique. En illustration, on considère quelques exemples.

ANALYTISCHE METHODEN BEI KONVEKTIVER WÄRMEÜBERTRAGUNG FÜR KONJUGIERTE PROBLEME

Zusammenfassung—Die vorliegende Arbeit berichtet über Lösungsmethoden für konvektive Wärmeübertragungsprobleme unter Berücksichtigung der Wärmeausbreitung in einem festen Körper der in Kontakt mit einer Flüssigkeitsströmung ist.

Im Einzelnen wird die Wärmeübertragung in einem zylindrischen und ebenen Rohr bei laminarer Strömung und bei Berücksichtigung der Reibung behandelt. Ausserdem wird sowohl das stationäre wie instationäre Wärmeübertragungsproblem für eine ebene Platte bei kompressibler Strömung untersucht. In allen Fällen wird der Wärmestrom in der Flüssigkeit mit dem Wandwärmestrom des festen Körpers verglichen und diskutiert.

Mit der analytischen Lösung wird ein neues Kriterium, das den Einfluss der thermo-physikalischen Eigenschaften der Wand auf den Wärmestrom charakterisiert, eingeführt. Das Problem wird an einigen Beispielen erläutert.